Quantum Graph Homomorphisms

An Operator Algebraic Approach

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Games on Graphs and Random Strategies

Non-Local (Two-prover One-round) Games

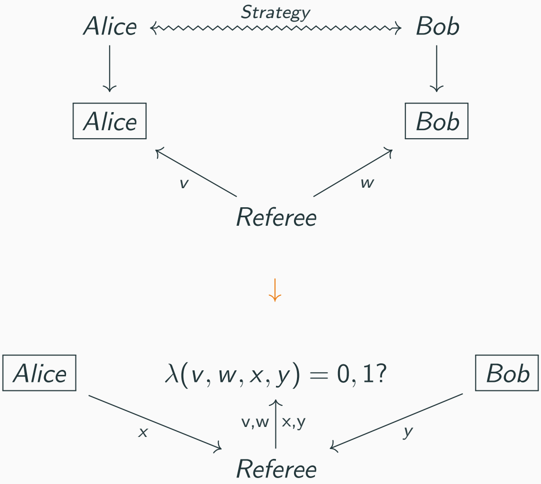
Let's play a game:
Suppose two non-communicating players, Alice and Bob, each receives an input from some finite set I and each must produce an output belonging to some finite set O.

The "rules" of the game are given by a function

$$\lambda : I \times I \times O \times O \to \{0, 1\}$$

where $\lambda(v, w, x, y) = 0$ means that if Alice and Bob receive inputs v, w, respectively, then producing respective outputs x, y is "disallowed".

Non-Local (Two-prover One-round) Games



Winning Strategies

A **strategy** for such a game is a conditional probability density p where $p(x, y|v, w)$ represents the probability that if Alice receives input v and Bob receives input w, then they produce outputs x and y, respectively.

Such a strategy is called **winning** or **perfect** provided:

$$\lambda(v, w, x, y) = 0 \implies p(x, y|v, w) = 0.$$

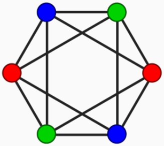
We call p **synchronous** if $p(x, y|v, v) = 0, \ \forall x \neq y$.

The Graph Colouring Game

Let $G = (V, E)$ be a graph with vertex set V and edges $E \subset V \times V$, the inputs are $I = V$ and the outputs O are a set of colors. The rules are that,

- $\lambda(v, v, x, y) = 0, \forall v \in V, \forall x \neq y$
- $\lambda(v, w, x, x) = 0, \forall (v, w) \in E, \forall x \in O$

Example



- $I = \{1, 2, 3, 4, 5, 6\}$
- $O = \{\text{Blue}, \text{Green}, \text{Red}\}$
- Strategy: Vertex Coloring
- Rules? ✓

The Graph Homomorphism Game

Given graphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$, a **graph homomorphism** is a mapping $f : V(G) \to V(H)$ such that if

$$(v, w) \in E(G) \implies (f(v), f(w)) \in E(H)$$

When a graph homomorphism from G to H exists we write $G \to H$. For the game, the inputs are $V(G)$ and the outputs are $V(H)$ and the rules are that,

- $\lambda(v, v, x, y) = 0$, $\forall v \in V(G)$, $\forall x \neq y$
- $\lambda(v, w, x, y) = 0$, $\forall (v, w) \in E(G)$, $\forall (x, y) \notin E(H)$

Coloring is a special case of homomorphism where $H = K_c$, $c = |O|$ is the number of colors.

Classical Strategies

A density p is called a **local** or **classical correlation** if there is a probability space (Ω, μ) and random variables,

$$f_v, g_w : \Omega \to O \text{ for each } v, w \in I$$

such that,

$$p(x, y|v, w) = \mu(\{\omega \mid f_v(\omega) = x, g_w(\omega) = y\})$$

Quantum Strategy

A density p is called a **quantum correlation** if it arises as follows:

Suppose Alice and Bob have finite dimensional Hilbert spaces $\mathcal{H}_A$, $\mathcal{H}_B$ and for each input $v \in I$ Alice has projective measurements (PVMs) $\{F_{v,x}\}_{x \in O}$ on $\mathcal{H}_A$, i.e., an $|O|$-outcome quantum experiment, and for each input $w \in I$ Bob has projective measurements $\{G_{w,y}\}_{y \in O}$ on $\mathcal{H}_B$ and they share a state $\psi \in \mathcal{H}_A \otimes \mathcal{H}_B$, then

$$p(x, y|v, w) = \langle F_{v,x} \otimes G_{w,y} \psi, \psi \rangle$$

This is the probability of getting outcomes x, y given that they conducted experiments v, w.

Quantum Commuting Strategy

A density p is called a **quantum commuting correlation** if there is a single Hilbert space $\mathcal{H}$, such that for each $v \in I$ Alice has projective measurements $\{F_{v,x}\}_{x \in O}$ on $\mathcal{H}$ and for each $w \in I$ Bob has projective measurements $\{G_{w,y}\}_{y \in O}$ on $\mathcal{H}$ satisfying,

$$F_{v,x} G_{w,y} = G_{w,y} F_{v,x}, \ \forall v, w, x, y$$

and

$$p(x, y | v, w) = \langle F_{v,x} G_{w,y} \psi, \psi \rangle$$

where $\psi \in \mathcal{H}$ is a shared state.

Sets of Correlations

When $|I| = n$ and $|O| = m$, we let:

- $C_{loc}(n, m)$ denote the set of all densities that are local correlations
- $C_q(n, m)$ denote the set of all densities that are quantum correlations
- $C_{qc}(n, m)$ denote the set of all densities that are quantum commuting correlations

Remark

Note that we can view each density p as a n^2m^2-tuple where each values is given by $p(x, y|v, w)$.

Conjectures of Connes and Tsirelson

Connes and Tsirelson

Set $C_{qa}(n,m) := \overline{C_q(n,m)}$. Here is what is known and why these objects are interesting:

- $C_{loc}(n,m) \subseteq C_q(n,m) \subseteq C_{qa}(n,m) \subseteq C_{qc}(n,m)$
- $C_{loc}(n,m)$ and $C_{qc}(n,m)$ are closed
- "Bounded Entanglement Conjecture": Is $C_q(n,m) = C_{qa}(n,m)\ \forall n,m$?
- Tsirelson conjecture (1993, 2006): Is $C_q(n,m) = C_{qc}(n,m)\ \forall n,m$?
- Ozawa (2012): Connes' embedding conjecture (1976) is true iff $C_{qa}(n,m) = C_{qc}(n,m),\ \forall n,m$
- Dykema-Paulsen (2015): Connes' embedding conjecture is true iff $C^s_{qa}(n,m) = C^s_{qc}(n,m),\ \forall n,m$

Remark

Notice that Tsirelson conjecture implies Connes' embedding conjecture.

Quantum Chromatic Numbers

Motivation

Question

PSSTW (2014): Can we distinguish these sets of correlations by studying existence of winning strategies for highly combinatorial games? Or conversely provide some evidence for the truth of these conjectures by showing no difference in existence?

Quantum Chromatic Numbers

Let G be a graph on n vertices. Recall that the chromatic number of a graph,

$$\chi(G) = min\{m : G \to K_m\}$$

Definition

For $t \in \{loc, q, qa, qc\}$, we define the **quantum chromatic numbers** by

$$\begin{aligned} \chi_t(G) = min\{m : \exists\ p(x, y|v, w) \in C_t(n, m),& \\ p(x, y|v, v) = 0\ \forall x \neq y,& \\ (v, w) \in E(G) \implies p(x, x|v, w) = 0\}.& \end{aligned}$$

Quantum Chromatic Numbers: Results

- $\chi_{loc}(G) \geq \chi_q(G) \geq \chi_{qa}(G) \geq \chi_{qc}(G)$
- Paulsen-Todorov (2013): $\chi_{loc}(G) = \chi(G)$
- For a Hadamard graph Ω_N, $(1.98)^N \leq \chi(\Omega_N) \leq 2^N$ and $\chi_q(\Omega_N) = \chi_{qa}(\Omega_N) = \chi_{qc}(\Omega_N) = N$
- For $\chi_t(G) \leq m$:

	χ_{loc}	χ_q	χ_{qa}	χ_{qc}
Complexity	NP-Complete	NP-Hard	?	?
Algorithm	SDP	?	?	SDP

- RM (2012): Analog of the fractional chromatic number for χ_q
- PSSTW (2014): Analog of the fractional chromatic number for χ_{qc}

Quantum Graph Homomorphisms

Quantum Graph Homomorphisms

For $t \in \{loc, q, qa, qc\}$, we write $G \xrightarrow{t} H$ provided that there exist a $p(x, y|v, w) \in C_t(n, m)$ such that,

- $p(x, y|v, v) = 0, \ \forall v \in V(G), \forall x \neq y$
- $p(x, y|v, w) = 0, \ \forall (v, w) \in E(G), \forall (x, y) \notin E(H)$

i.e., if there exists a winning t-strategy for the graph homomorphism game. We call these **quantum graph homomorphisms**.

Quantum Graph Homomorphisms: Results

Theorem (O-Paulsen)

Let G and H be graphs.

- $G \to H \iff G \stackrel{loc}{\to} H$
- $G \longrightarrow H \implies G \stackrel{q}{\longrightarrow} H \implies G \stackrel{qa}{\longrightarrow} H \implies G \stackrel{qc}{\longrightarrow} H$
- $G \stackrel{t}{\to} H$ *and* $H \stackrel{t}{\to} K$ *implies* $G \stackrel{t}{\to} K$
- $\chi_t(G) = min\{m : G \stackrel{t}{\to} K_m\}$

C*-algebras and Graph Homomorphisms

C*-Homomorphisms

Given graphs G and H, we write $G \overset{C^*}{\to} H$ if we can find projections $\{E_{v,x} : v \in V(G), x \in V(H)\}$ on some Hilbert space $\mathcal{H}$ satisfying:

- $E_{v,x}E_{v,y} = 0, \ \forall x \neq y$
- $\sum_x E_{v,x} = I, \ \forall v$
- If $(v, w) \in E(G)$ and $(x, y) \notin E(H) \implies E_{v,x}E_{w,y} = 0$

If $G \overset{C^*}{\to} H$ exists, we let $\mathcal{A}(G, H)$ denote the "universal" unital C*-algebra generated by $\{E_{v,x} : v \in V(G), x \in V(H)\}$.

C*-Homomorphisms: Results

Theorem (O-Paulsen)

$$G \longrightarrow H \implies G \xrightarrow{q} H \implies G \xrightarrow{qa} H \implies G \xrightarrow{qc} H \implies G \xrightarrow{C^*} H$$

Theorem (O-Paulsen)

Let G and H be graphs.

- $G \to H \iff \mathcal{A}(G,H)$ *has a 1-dimensional representation*
- $G \stackrel{q}{\to} H \iff \mathcal{A}(G,H)$ *has a finite dimensional representation*
- $G \stackrel{qc}{\to} H \iff \mathcal{A}(G,H)$ *has a trace*

Don't have a characterization of $G \stackrel{qa}{\to} H$.

C*-Homomorphisms: Results

Theorem (O-Paulsen)

Assume Tsirelson's conjecture is true. Then $\mathcal{A}(G, H)$ has a trace iff $\mathcal{A}(G, H)$ has a finite dimensional representation.

Theorem (O-Paulsen)

Let G be a graph.

- $\chi(G) \leq m \iff \mathcal{A}(G, K_m)$ *has a 1-dimensional representation*
- $\chi_q(G) \leq m \iff \mathcal{A}(G, K_m)$ *has a finite dimensional representation*
- $\chi_{qc}(G) \leq m \iff \mathcal{A}(G, K_m)$ *has a trace*

Connections with Complexity Theory

Theorem (O-Paulsen)

Let G be a graph on n vertices and fix m.

- *There exist algorithms for deciding if $\mathcal{A}(G, K_m)$ has a 1-dimensional representation, but the problem of deciding if $\mathcal{A}(G, K_m)$ has a 1-dimensional representation is NP-complete.*
- *The problem of deciding if $\mathcal{A}(G, K_m)$ has a finite dimensional representation is NP-hard and currently there is no known algorithm.*
- *There exists an SDP that decides if $\mathcal{A}(G, K_m)$ has a trace.*

C*-algebras and Chromatic Numbers

C*-Chromatic Number

Now that we have a "new" notion of a graph homomorphism we can define,

$$\chi_{c^*}(G) = min\{m : G \stackrel{C^*}{\to} K_m\},$$

Theorem

- $\chi_{qc}(G) \geq \chi_{c^*}(G)$
- $\vartheta(\overline{G}) \leq \chi_{c^*}(G)$
- $\chi_{c^*}(G \square H) = max\{\chi_{c^*}(G), \chi_{c^*}(H)\}$
- *Defined analog of the fractional chromatic number for $\chi_{c^*}(G)$:*

$$\xi_{c^*}(G) \leq \chi_{c^*}(G)$$

Quantum Cores

Core of a Graphs

Definition

A retract of a graph G is a subgraph H of G such that there exists a homomorphism $\phi : G \to H$, called a retraction, with $\phi(x) = x$ for any $x \in V(H)$. A **core** is a graph which does not retract to a proper subgraph.

Existence of Cores

Theorem

Let ϕ *be an endomorphism of a graph* G*. Then there is an* n *such that* $R = \phi^n(G)$ *is a retract of* G *(and* ϕ^n *a retraction). Hence, every graph has a core.*

“Quantum” Core: Motivation

We will now try to address a question asked by Roberson in his thesis: how should we define a “quantum” core of a graph? We will use the above theorem as a guiding principle to define what we call “quantum cores”.

Ingredients for a “Quantum” Core

- You need to be able to define some kind of “generalized retraction”.
- You need to be able to “iterate” to get a “retract”.
- You need to be able to talk about “minimality” in order to defined a “core”.

Quantum Homomorphism as CP maps

Definition

Let $p(x, y|v, w) \in Q_x(n, m)$ be a winning x-strategy, let $E_{v,w} \in M_n$ and $E_{x,y} \in M_m$ denote the canonical matrix unit bases.

Define the map $\phi_p : M_n \to M_m$ by

$$\phi_p(E_{v,w}) = \sum_{x,y} p(x, y|v, w) E_{x,y}$$

We say that p **implements** the map ϕ_p.

The Operator System of Graph

We define the **operator system of the graph** G to be

$$S_G := Span\{E_{ij} : (i,j) \in E(G) \text{ or } i = j\} \subset M_n(\mathbb{C})$$

Theorem

G_1 is isomorphic to G_2 $\iff$ S_{G_1} is unitally, completely order isomorphic to S_{G_2}

Properties of ϕ_p

Theorem

- *ϕ_p is CP.*
- *$\phi_p(S_G) \subseteq S_H$.*
- *ϕ_p is trace-preserving on S_G.*

Composition of ϕ_p's

Theorem

Let $x \in \{l, q, qa, qc, vect\}$, let $p(x, y|v, w) \in Q_x(n, m)$ and let $q(a, b|x, y) \in Q_x(m, l)$. Then

$$r(a, b|v, w) := \sum_{x,y} q(a, b|x, y)p(x, y|v, w) \in Q_x(n, l).$$

Moreover, if p and q are winning x-strategies, then r is a winning x-strategy.

Theorem

If $\phi_p : M_n \to M_m$, $\phi_q : M_m \to M_l$ and $\phi_r : M_n \to M_l$ are the corresponding linear maps, then $\phi_r = \phi_q \circ \phi_p$.

Relation to the Lovász Theta Function

Theorem

Let G and H be graphs on n and m vertices, respectively. Let $p(x, y|v, w) \in Q_x(n, m)$ be a winning x-strategy for a graph homomorphism. Then $\|\phi_p\|_{cb} \leq \vartheta(G)$.

Recall that,

$$||\phi||_{cb} := \sup_{n \in \mathbb{N}}\{||\phi \otimes I_n||\}.$$

Banach Limits

A Banach generalized limit, is a positive linear functional f on $\ell^\infty(\mathbb{N})$, such that:

- If $(a_k) \in \ell^\infty(\mathbb{N})$ and $\lim_k a_k$ exists,then $f((a_k)) = \lim_k a_k$.
- If $b_k = a_{k+1}$, then $f((b_k)) = f((a_k))$.
- If $(a_k) \in \ell^\infty(\mathbb{N})$ and $a_k \geq 0$, then $f((a_k)) \geq 0$.

The existence and construction of these is presented in Conway's A Course in Functional Analysis, along with many of their other properties. Often a Banach generalized limit functional is written as *glim*.

Generalized Retraction: Construction

Let G be a graph on n vertices, let $x \in \{loc, qa, qc, vect\}$, and let $p(x, y|v, w) \in C_x(n, n)$ be a winning x-strategy. Now:

1. Set $p_1(x, y|v, w) := p(x, y|v, w)$
2. Set $p_{k+1}(x, y|v, w) := \sum_{a,b} p(x, y|a, b)p_k(a, b|v, w)$
3. Define $r(x, y|v, w) := glim\big(p_k(x, y|v, w)\big)$,

Theorem

Following properties hold for r:

- $r(x, y|v, w) \in C_x(n, n)$ *is a winning x-strategy.*
- $r(x, y|v, w) = \sum_{a,b} r(x, y|a, b)r(a, b|v, w)$.

Generalized Retraction: Properties

Theorem

Let $p(x, y|v, w) \in C_x(n, n)$ *be a winning x-strategy and let* $\psi_p : M_n \to M_n$ *be the map implemented by the above r (i.e.* $\psi_p := \phi_r$ *) Then:*

- ψ_p *is CP and TP on* S_G.
- $\psi_p \circ \phi_p = \phi_p \circ \psi_p = \psi_p$
- $\psi_p \circ \psi_p = \psi_p$

Minimality

Theorem

Let G be a graph on n vertices and let $x \in \{loc, qa, qc, vect\}$. Then there exists $r(x, y|v, w) \in C_x^s(n, n)$ implementing a map $\phi_r : M_n \to M_n$ that is idempotent and minimal in the partial order on idempotent maps of the form ϕ_p implemented by a quantum x-homomorphism of G.

Quantum Cores

Definition

Let G be a graph on n vertices and let $x \in \{loc, qa, qc, vect\}$. Then a **quantum** x-**core** for G is any $p(x, y|v, w) \in C_x(n, n)$ that implements a quantum x-homomorphism such that ϕ_p is idempotent and minimal among all ϕ_p implemented by a quantum x-homomorphism of G.

Open Problems

- Give a characterization for $G \stackrel{qa}{\to} H$.
- Complexity level of determining if $\mathcal{A}(G, K_m)$ exists, i.e. $\chi_{c^*}(G) \leq m$.
- Prove theorems about our "quantum" core.
- If we could show that whenever families of projections on an infinite dimensional Hilbert space exist that satisfy the relations for $\mathcal{A}(G, K_m)$ to exist, then these relations could be met by projections on a finite dimensional space, then

$$\chi_q(G) = \chi_{qa}(G) = \chi_{qc}(G) = \chi_{c^*}(G).$$

References

All on ArXiv:

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- V. I. Paulsen, S. Severini, D. Stahlke, I. G. Todorov, A. Winter, *Estimating quantum chromatic numbers*
- V. I. Paulsen, I. G. Todorov, *Quantum chromatic numbers via operator systems*

¡Gracias por su atención!