Lovász Theta Type Norms

Graphs, SDPs, and Operator Systems

Carlos Ortiz
University of Houston

Based on joint work with Vern Paulsen

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Introduction

Operator System

Definition

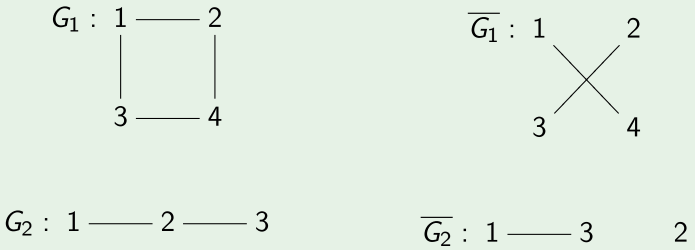
We define an operator system S to be any unital $*$-closed subspace of $B(H)$. Let S^+ denote the cone of positive elements in S. Unless otherwise stated, all operator system are assumed to be finite dimensional.

Inner product in M_n

Recall that M_n is a Hilbert space with respect to the inner product $\langle a, b \rangle = tr(ab^*)$, $a, b \in M_n$. Thus, given any subspace $\mathcal{S} \subseteq M_n$, one may form the orthogonal complement $\mathcal{S}^\perp$ of $\mathcal{S}$.

Graphs

Example



The Operator System of a Graph

Definition

We define the operator system of the graph G to be the operator system $S_G := \{(I + A_G) \circ X : X \in M_n\}$, where A_G is the adjacency matrix of G.

Recall that with the above inner product, the orthogonal projection $P_G : M_n \to S_G$ is given $P_G(X) = (I + A_G) \circ X$. It is not too hard to see that,

$$\boxed{S_G^{\perp} = \{A_{\overline{G}} \circ X : X \in M_n\}}$$

The Operator System of a Graph

Example

$$G:\ 1 \text{ —— } 2 \text{ —— } 3$$

$$S_G = \begin{pmatrix} \bullet & \bullet & 0 \\ \bullet & \bullet & \bullet \\ 0 & \bullet & \bullet \end{pmatrix}$$

$$S_G^{\perp} = \begin{pmatrix} 0 & 0 & \bullet \\ 0 & 0 & 0 \\ \bullet & 0 & 0 \end{pmatrix}$$

Motivation

Motivation

Operator systems have always played a fundamental role in quantum information theory e.g. the model for a quantum channel is a completely positive trace preserving map. In 2010, Duan, Severini, and Winter proposed,

$$\text{Operator System Theory} = \text{“Noncommutative Graph Theory”}.$$

Motivation

In addition,

Theorem

G is isomorphic to $H \iff S_G$ is completely order isomorphic to S_H

Motivation

- If operator systems are like "generalized graphs", then can we apply operator system theory to classical graph theory and vice versa?
- Does the extra structure of the operator system tells us anything new about the graph?

Lovász "magic" number

In 1979, László Lovász published a paper where he defines a number that only depended on G, usually denoted by $\vartheta(G)$, with some interesting properties,

$$\underset{\textbf{NP}-\textbf{Hard}}{\alpha(\overline{G})} \leq \underset{\boxed{\textbf{SDP}}}{\vartheta(\overline{G})} \leq \underset{\textbf{NP}-\textbf{Hard}}{\chi(G)}$$

SDP: Semidefinite Program. These are solvable in polynomial time to within an additive error $\epsilon > 0$.

$$\boxed{\vartheta(G \boxtimes H) = \vartheta(G) \cdot \vartheta(H)}$$

(This product satisfies $S_{G \boxtimes H} = S_G \otimes S_H$)

Operator Systems?

It turns out that the two canonical quotient norms you can define on a operator system behave just like the Lovász theta function.

Graph Parameters via Quotient Norms

Quotient Norms

Given an operator system S and a kernel $\mathcal{J} \subseteq \mathcal{S}$; that is, a kernel of some unital completely positive map $\phi : \mathcal{S} \to \mathcal{B}(H)$ for some Hilbert space H, there is a way of defining two different quotient structures on $S/\mathcal{J}$.

- Operator/norm space quotient
- Operator system quotient

From these two structures, we can define two norms for any $X \in S$, which we will denote by $||X + \mathcal{J}||_{osp}$ and $||X + \mathcal{J}||_{osy}$.

Generalized Lovász Function

If we define,

$$\vartheta(\mathcal{J}) = \sup\{\|1 + a\| : a \in \mathcal{J}, 1 + a \in S^+\}$$

Theorem

$$||X + \mathcal{J}||_{osy} \leq ||X + \mathcal{J}||_{osp} \leq \vartheta(\mathcal{J}) \cdot ||X + \mathcal{J}||_{osy}$$

Generalized Lovász Function

Theorem

Let G be a graph on k vertices. Then $\mathcal{S}_G^{\perp}$ is a kernel in M_k. In addition,

$$\vartheta(S_G^{\perp}) = \vartheta(G)$$

Notice that this parameter only depends on the kernel. This means that...

Two New Families of Parameters

...we can define the following two families of parameters,

$$\sigma(G, X) := ||X + S_G^{\perp}||_{osy}$$

$$d_{\infty}(G, X) := ||X + S_G^{\perp}||_{osp}$$

Result

Theorem

For $X \in M_n$ *and* $Y \in M_m$*,*

- $d_\infty(G \boxtimes H, X \otimes Y) = d_\infty(G, X) \cdot d_\infty(H, Y)$
- $\sigma(G \boxtimes H, X \otimes Y) = \sigma(G, X) \cdot \sigma(H, Y)$

Theorem

For $X \in M_n(\mathbb{R})$*,*

- $d_\infty(G, X)$ *has an SDP representation satisfying strong duality.*
- $\sigma(G, X)$ *has an SDP representation satisfying strong duality.*

$d_\infty(G, I + A_G)$ vs. $\vartheta(\overline{G})$

Recall that,

$$d_\infty(G, I + A_G) = \min\{||(I + A_G) + K|| : K \in S_G^\perp\}$$

$$\vartheta(\overline{G}) = \min\{\lambda_1([I + A_G] + K) : K \in S_G^\perp\}$$

hence,

$$\vartheta(\overline{G}) \leq d_\infty(G, I + A_G)$$

Question

Is

$$\vartheta(\overline{G}) = d_\infty(G, I + A_G)?$$

No!

A new condition on G

Theorem

If $\vartheta(\overline{G}) = d_\infty(G, I + A_G)$, *then* $\frac{\chi(G)}{||P_G||} \leq \vartheta(\overline{G}) \leq \chi(G)$

Open Problems

- Find necessary and sufficient conditions such that $d_\infty(G, I + A_G) = \vartheta(\overline{G})$.
- Does $||X + S_G^\perp||_{osp/osy}$ tell us anything else about G?

Our Paper

Lovász Theta Type Norms and Operator Systems, preprint, arXiv: **1412.7101**

Thanks!