Non-Local Games on Graphs

An Operator Algebraic Approach

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Graph Homomorphism Game

Graph Homomorphism

Given graphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$, a **graph homomorphism** is a mapping $f : V(G) \to V(H)$ such that if

$$(v, w) \in E(G) \implies (f(v), f(w)) \in E(H)$$

When a graph homomorphism from G to H exists we write $G \to H$.

Graph Homomorphism (Non-Local) Game

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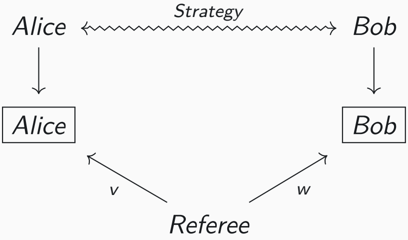
The "rules" of the game are given by a function

$$\lambda : V(G) \times V(G) \times V(H) \times V(H) \to \{0,1\}$$

such that,

- $\lambda(v,v,x,y) = 0, \forall v \in V(G), \forall x \neq y$
- $\lambda(v,w,x,y) = 0, \forall (v,w) \in E(G), \forall (x,y) \notin E(H)$

Graph Homomorphism Game



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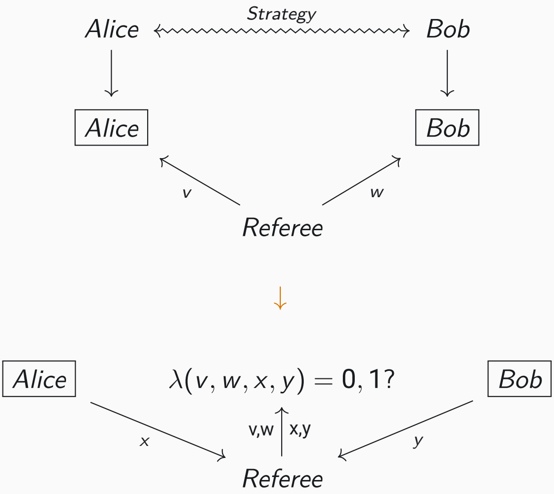

A **strategy** for such a game is a conditional probability density p where $p(x, y|v, w)$ represents the probability that if Alice receives vertex v and Bob receives vertex w, then they produce vertices x and y, respectively.

Winning Strategies

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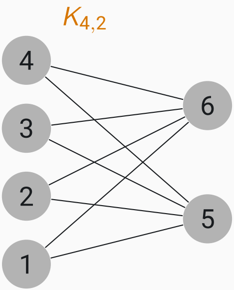
Such a p is a **winning strategy** provided:

$$\lambda(v, w, x, y) = 0 \implies p(x, y|v, w) = 0.$$

Graph Homomorphism

Example



f

$1 \to 1$

$2 \to 1$

$3 \to 1$

$4 \to 1$

$5 \to 2$

$6 \to 2$

Strategy: $p(x, y|v, w) = \begin{cases} 1 & \text{if } f(v) = x, f(w) = y \\ 0 & \text{otherwise} \end{cases}$

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Sets of Correlations

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- Ozawa (2012): Connes' embedding conjecture (1976) is true iff $\overline{C_q(n, m)} = C_{qc}(n, m), \forall n, m$

Quantum Graph Homomorphisms

For $t \in \{loc, q, qc\}$, we write $G \xrightarrow{t} H$ provided that there exist a winning strategy $p(x, y|v, w) \in C_t(n, m)$ for the homomorphism game. We call these **quantum graph homomorphisms**.

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- $G \xrightarrow{t} H$ *and* $H \xrightarrow{t} K$ *implies* $G \xrightarrow{t} K$

C*-algebras and Graph Homomorphisms

C*-Homomorphisms

Given graphs G and H, we write $G \overset{C^*}{\to} H$ if we can find projections $\{E_{v,x} : v \in V(G), x \in V(H)\}$ on some Hilbert space $\mathcal{H}$ satisfying:

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    - - *There exist an SDP (PSSTW, 2014)*

Reference

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On ArXiv:

C. Ortiz, V. I. Paulsen, *Quantum graph homomorphisms via operator systems*

Other topics:

- Quantum chromatic numbers
- Quantum core of a graph

Thanks!