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Classification of Rank 5 Premodular Categories

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Joint work with Paul Bruillard

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Overview

1. Motivation
2. Classification
3. Categories
4. Results

Motivation

What is Quantum Computation?

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Definition [Freedman, Kitaev, Larsen, Wang '03]

Quantum Computation is any computational model based upon the theoretical ability to manufacture, manipulate, and measure quantum states

What is Topological Quantum Computation?

Definition [Freedman, Kitaev, Larsen, Wang '03]

Topological Quantum Computation is any computational model based upon the theoretical ability to manufacture, manipulate, and measure quantum states using topological phases of matter.

Topological Quantum Field Theories

Definition [Nayak, et al ’08]

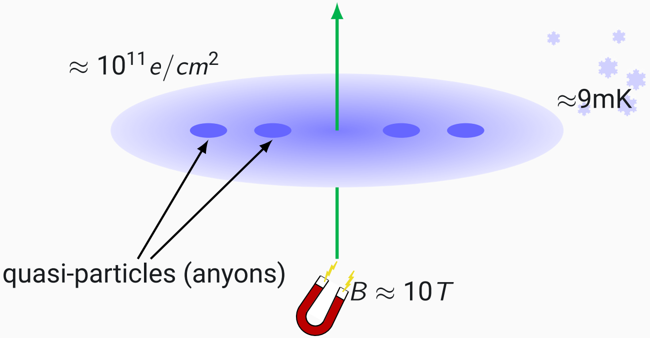
A topological phase of a matter (TPM) is a physical system such that its low-energy effective field theory* is described by a TQFT.

Definition [Witten, et al ‘88]

A topological quantum field theory (TQFT) is quantum mechanical model where “amplitudes only depend on the topology of the process”†.

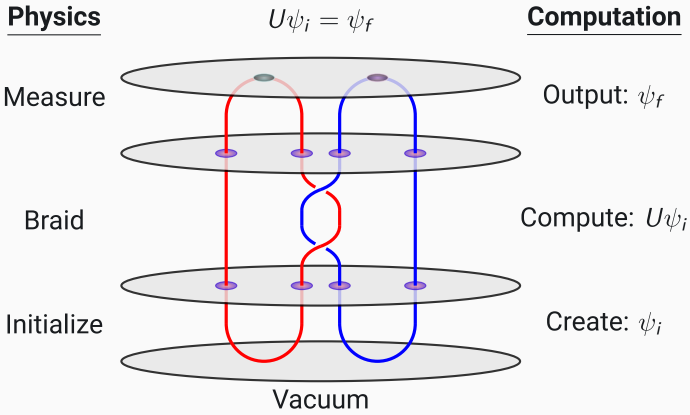
*...system is away from any boundary and has low energy and temperature.
†Local perturbations will not change the state of your system.

Example: Two-Dimensional Electron Gas



- These things exist! (e.g. $GaAs$, α-$RuCl_3$, $YbMgGaO_4$)
- There is theoretical (and some experimental) evidence that you can perform quantum computation with some of these phases.
- Nobel prizes: experimental (1985, 1998) and theoretical (2016).

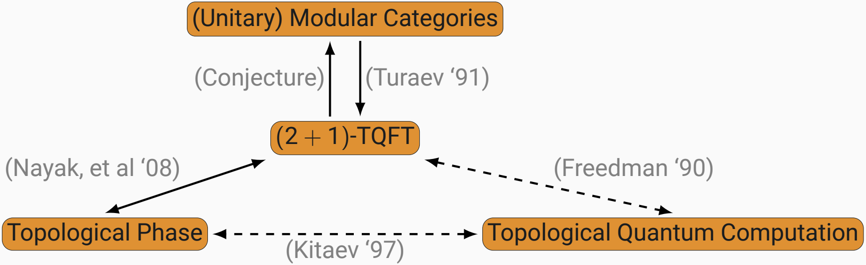
Computational Model



- Performing gate operations = Braiding particles
- Computation is topologically protected from decoherence.

Mathematical Structure

The appropriate mathematical structure is a modular category.



Morally, a classification of modular categories gives you a classification of topological phases.

Classification

Classification by rank

Theorem [Bruillard, Ng, Rowell, Wang '13]

There are finitely many modular categories of a given rank‡ r.

- Complete classification up to rank 5.

Conjecture

There are finitely many premodular§ categories of a given rank r.

- Complete classification up to rank 4 **[Bruillard]**.

‡Number of different particles in your theory.
§...these are thought to be useful for (3+1) TQFTs.

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The classification problem is hard!

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Problems

- No general classification technique.
- Requires a lot of result from various areas of math on a case-by-case basis.
- In order to understand modular categories, you need to understand premodular categories, and vice versa.

Categories

Representation Category of a Group

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$$\mathrm{Rep}(G)$$

Basic properties:

- $(\mathrm{Rep}(G), \oplus, \otimes, *)$
- $Hom^G(\rho, \varphi)$ is a finite dimensional vector space
- $|Irr(G)| < \infty$
- $\phi = \bigoplus_k \alpha_k \psi_k, \psi_k \in Irr(G)$

Premodular Categories

Definition

A Premodular category is a spherical, braided, fusion category.

- Abelian Monodial Category $(\mathcal{C}, \oplus, \otimes)$
- $\mathbb{C}$-linear: $Hom(X, Y)$ is a finite dimensional vector space
- Finite rank: finite number of simple classes $\{X_0 = \mathbb{1}, X_1, ..., X_n\}$
- Semisimple: $X \cong \bigoplus_k \mu_k X_k$
- Dual object: X^*
- $c_{X,Y} : X \otimes Y \cong Y \otimes X$
- Canonical notion of $Tr_{\mathcal{C}}$

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Categorical Data

These set of axioms give rise to data that is an invariant for categories,

- $S := (\ Tr_{\mathcal{C}}(c_{X,Y^*} c_{Y^*,X})\ )$
- $\theta_X :=$ root of unity **[Vafa '88]**

Definition

If $\mathcal{C}$ is premodular and $Det(S) \neq 0$, we say $\mathcal{C}$ is a modular category.

Mathematical Importance of Fusion Categories

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We can think of the theory of fusion categories as an extension of representation theory:

Theorem [Deligne, Milne '82]

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Remark

You get modular categories from von Neumann Algebras, vertex operator algebras, Hopf algebras, and Quantum Groups.

Results

Theorem

If $\mathcal{C}$ is a rank 5 premodular category,¶

- $\mathcal{C}$ is symmetric and is given by $\mathrm{Rep}(G, z)$ where G is $\mathbb{Z}_5$, D_8, Q_8, D_{14}, $\mathbb{Z}_5 \rtimes \mathbb{Z}_4$, $\mathbb{Z}_7 \rtimes \mathbb{Z}_3$, $\mathfrak{S}_4$, or $\mathfrak{A}_5$, and z is a central element of order 2.
- $\mathcal{C}$ is properly premodular and Grothendieck equivalent to:
  - $PSU(2)_8$ with $\mathcal{C}' = \mathrm{Rep}(\mathbb{Z}_2)$
  - $\mathrm{Rep}(D_8)$ with $\mathcal{C}' = \mathrm{Rep}(\mathbb{Z}_2)$
  - $\mathrm{Rep}(D_{14})$ with $\mathcal{C}' = \mathrm{Rep}(\mathbb{Z}_2)$
  - $\mathrm{Rep}(\mathfrak{S}_4)$ with $\mathcal{C}' = \mathrm{Rep}(\mathfrak{S}_3)$
- $\mathcal{C}$ is modular and it is Grothendieck equivalent to $SU(2)_4, SU(2)_9/\mathbb{Z}_2, SU(5)_1$, or $SU(3)_4/\mathbb{Z}_3$ **[BNRW '15]**.

Moreover, we found the categorical data for each case.

¶... well ... technically, pseudo-unitary premodular...

Outline of Proof

Suppose $\mathcal{C}$ is a rank 5 premodular category...

1. Consider the subcategory, $\mathcal{C}' := \{Y \,|c_{X,Y}c_{Y,X} = id_{X \otimes Y}, X \in \mathcal{C}\}$. Since $\mathcal{C}'$ is symmetric, it looks like $\mathrm{Rep}(G)$ [**Deligne '02**].
2. Relate $\mathcal{C}$ to a modular category using a construction called de-equivariantization [**Bruguières '00**].
3. Exploit the structure of the modular category to deduce information about $\mathcal{C}$ [**Burciu, Natale '13**].
4. Ad-hoc methods...

References

On ArXiv:

P. Bruillard, C. Ortiz, *Rank 5 premodular categories*

Thanks!